

Non linear difference equations arising from a deformation of the q -Laguerre weight.

Yang Chen

Department of Mathematics

University of Macau, (yangbrookchen@yahoo.co.uk)

Macau, China

James Griffin

Department of Mathematics

American University of Sharjah, (jamescgriffin@gmail.com)

April 14, 2014

Abstract

We study, in this paper, a one parameter deformation of the q -Laguerre weight function. An investigation is made on the polynomials orthogonal with respect to such a weight. With the aid of the two compatibility conditions previously obtained in [8] and the q -analog of a sum rule obtained in this paper, we derive expressions for the recurrence coefficients in terms of certain auxiliary quantities, and show that these quantities satisfy a pair of first order non linear difference equations. These difference equations are similar in form to the recognized asymmetric discrete Painleve systems such as αq -P-IV and αq -P-V.

1 Introduction

In this paper we derive difference equations satisfied by the recurrence coefficients of a family of polynomials orthogonal with respect to the weight supported on $[0, \infty)$,

$$w(x, \alpha, t; q) = \frac{x^\alpha}{(- (1 - q)x; q)_\infty (- (1 - q)\frac{t}{x}; q)_\infty}, \quad t \geq 0, \quad \alpha > -1, \quad 0 < q < 1 \quad (1.1)$$

where

$$(a; q)_\infty := \prod_{j=0}^{\infty} (1 - a q^j).$$

In the special case of $q \rightarrow 1^-$ this reduces to

$$x^\alpha e^{-x} e^{-t/x},$$

first considered by Chen and Its [11], which is a singular deformation of the ordinary Laguerre weight. It was shown in that case that the log-derivative of the Hankel determinant (with respect to t) is the τ -function of a particular P-III.

In the limit $t \rightarrow 0^+$ our weight reduces to the q -Laguerre weight introduced by Moak [18]. It transpires that for $t = q/(1-q)^2$ and $\alpha = 0$ the corresponding orthogonal polynomials are the Stieltjes-Wigert polynomials. For this value of t and $\alpha \neq 0$ the orthogonal polynomials were studied in [1] by Askey.

Such a q -deformation causes the weight to decrease slowly near ∞

$$w(x, 0, t; q) = O\left(e^{-c(\ln x)^2}\right),$$

where c is positive constant, and causes similar decrease near 0. Weights with such slow decrease near ∞ were investigated in the context of unitary matrix ensembles which arise from electron transport in disordered systems [12].

Let $\{P_n(x)\}$ be the monic polynomials orthogonal with respect to a weight w on the interval $[0, \infty)$. That is

$$\int_0^\infty P_n(x)P_m(x)w(x) dx = h_n \delta_{nm}, \quad (1.2)$$

where h_n is the square of the L^2 norm.

It is well known that the polynomials satisfy a three term recurrence relation,

$$xP_n(x) = P_{n+1}(x) + \alpha_n P_n(x) + \beta_n P_{n-1}(x). \quad (1.3)$$

We take the initial conditions to be $\beta_0 P_{-1}(x) = 0$ and $P_0(x) = 1$.

Our monic polynomial has a monomial expansion

$$P_n(x) = x^n + p(n)x^{n-1} + \dots \quad (1.4)$$

It is clear from the recurrence relation that

$$\alpha_n = p(n) - p(n+1), \quad (1.5)$$

and consequently

$$\sum_{j=0}^{n-1} \alpha_j = -p(n). \quad (1.6)$$

For the weight we study, although explicit formulae are not found, we show that the recurrence coefficients α_n and β_n , can be expressed in terms of the quantities x_n and y_n which are solutions of a pair of coupled difference equations akin to αq -P-IV and αq -P-V. In addition, we show that $p(n)$ satisfies a non linear second order difference equation in n .

In [10], Chen and Ismail showed that under suitable conditions on the weight w , the polynomials satisfied a first order structural relation with respect to the operator D_q , defined by,

$$(D_q f)(x) = \frac{f(x) - f(qx)}{x(1-q)}.$$

Specifically, they proved the following theorem.

Theorem 1.1 *Let*

$$A_n(x) = \frac{1}{h_n} \int_0^\infty \frac{u(qx) - u(y)}{qx - y} P_n(y) P_n(y/q) w(y) dy \quad (1.7)$$

and

$$B_n(x) = \frac{1}{h_{n-1}} \int_0^\infty \frac{u(qx) - u(y)}{qx - y} P_n(y) P_{n-1}(y/q) w(y) dy, \quad (1.8)$$

where

$$u(x) = -\frac{D_{q^{-1}}w(x)}{w(x)}. \quad (1.9)$$

Then the orthogonal polynomials satisfy the q -difference relation,

$$D_q P_n(x) = \beta_n A_n(x) P_{n-1}(x) - B_n(x) P_n(x). \quad (1.10)$$

The above theorem is a q -analog of the structural relation appearing in [8]. Furthermore, it was shown in [10] that the functions $A_n(x)$ and $B_n(x)$ satisfy the supplementary conditions

$$B_{n+1}(x) + B_n(x) = (x - \alpha_n) A_n(x) + x(q - 1) \sum_{j=0}^n A_j(x) - u(qx), \quad (qS_1)$$

and

$$\beta_{n+1} A_{n+1}(x) - \beta_n A_{n-1}(x) = 1 + (x - \alpha_n) B_{n+1}(x) - (qx - \alpha_n) B_n(x). \quad (qS_2)$$

Equations (qS_1) and (qS_2) are q -analogs of the supplementary conditions (S_1) and (S_2) appearing in [8].

If the function $u(x)$ is rational then so are the functions $A_n(x)$ and $B_n(x)$. By comparing coefficients on both sides of the supplementary conditions one can obtain non-linear difference equations satisfied by the recurrence coefficients. In certain cases these equations can be solved explicitly. For example, in [10] the recurrence coefficients for the q -Laguerre and Stieltjes-Wigert polynomials were extracted explicitly as a result of this procedure. In general it is not always possible to find exact solutions, and in those cases the non linear difference equations themselves become of interest mainly because they have been shown in many circumstances to be related to q -discrete Painleve equations. For example, in [5], Boelen and Van Assche considered the semi-classical q -Laguerre weight supported on $[0, \infty)$,

$$w(x) = \frac{x^\alpha (-p/x^2; q^2)_\infty}{(-x^2; q^2)_\infty (-q^2/x^2; q^2)_\infty}, \quad p \in [0, q^{-\alpha}), \quad \alpha \geq 0 \quad (1.11)$$

and found that the recurrence coefficients were related to a solution of the q -discrete Painleve V equation. For $p = 0$ the weight (1.11) is the same as that studied by Askey in [1] with the variable x replaced by x^2 and the recurrence coefficients in this case were found to be related to a particular q -Painleve III. A non-linear difference equation related to the following generalization of (1.11)

$$w(x) = \frac{x^\alpha (-p_1/x^2; q^2)_\infty (-p_2/x^2; q^2)_\infty}{(-x^2; q^2)_\infty (-q^2/x^2; q^2)_\infty}, \quad x \in [0, \infty), \quad p_1 p_2 < q^{2-\alpha}, \quad p_1 > 0, \quad p_2 > 0, \quad \alpha \geq 0$$

was studied in [13].

In [15] and [16], non-linear difference equations are derived in connection with q -analogs of the Freud weights. Also in [4], a modified q -Freud weight is shown to give rise to a q -difference equation related to the anti-symmetric αq -P-V equations.

In the paper of Chen and Its, [11], they made use of the differential-difference relation,

$$\frac{d}{dx}P_n(x) = \beta_n A_n(x)P_{n-1}(x) - B_n(x)P_n(x). \quad (1.12)$$

where $A_n(x)$ and $B_n(x)$ have the form

$$A_n(x) = \frac{1}{h_n} \int_0^\infty \frac{u(x) - u(y)}{x - y} P_n(y)P_n(y)w(y) dy \quad (1.13)$$

and

$$B_n(x) = \frac{1}{h_{n-1}} \int_0^\infty \frac{u(x) - u(y)}{x - y} P_n(y)P_{n-1}(y)w(y) dy, \quad (1.14)$$

with

$$u(x) = -\frac{w'(x)}{w(x)}.$$

The supplementary conditions in this case are

$$B_{n+1}(x) + B_n(x) = (x - \alpha_n)A_n(x) - u(x), \quad (S_1)$$

and

$$\beta_{n+1}A_{n+1}(x) - \beta_n A_{n-1}(x) = 1 + (x - \alpha_n)B_{n+1}(x) - (x - \alpha_n)B_n(x). \quad (S_2)$$

Chen and Its also made use of the following equation that can be thought of as the first integral of (S_1) and (S_2) ,

$$B_n^2(x) + u(x)B_n(x) + \sum_{j=0}^{n-1} A_j(x) = \beta_n A_n(x)A_{n-1}(x). \quad (S'_2).$$

A derivation of (S'_2) is given in [11] and the equation first appeared in [17]. In [11] they found a pair of coupled non-linear difference equations whose solutions were related to the recurrence coefficients. In this context they also found a particular Painleve-III differential equation in the parameter t . The equation (S'_2) appeared in [2] in connection with a Painleve V equation, in [7], in connection with a Painleve IV equation and in [3], in connection with a Painleve V equation.

For the weight appearing in (1.1), we make use of Theorem 1.1, as well as equations (qS_1) and (qS_2) in an attempt to find expressions for the recurrence coefficients in terms of solutions to a pair of non-linear difference equations. Observe that the quantity $\sum_j A_j(x)$ appears in (qS_1) and not in (qS_2) . We will find that in order to deal with this sum effectively we will require an additional equation involving this quantity. Therefore, instrumental in our approach is the derivation of a q -analog of the equation (S'_2) which can be thought of as

a first integral of (qS_1) and (qS_2) . This equation appears to be new. Note that for weights such as (1.11) the function $u(x)$ simplifies sufficiently that the quantities involving $\sum_j A_j(x)$ are eliminated without the use of another supplementary condition. This will not be the case for the weight (1.1).

The three main results of the paper are summarized below.

Theorem 1.2 *Let $A_n(x)$ and $B_n(x)$ be given by (1.7) and (1.8). Then*

$$\beta_n A_n(x) A_{n-1}(x) = B_n^2(x) + u(qx) B_n(x) + (1 + (1 - q)x B_n(x)) \sum_{j=0}^{n-1} A_j(x). \quad (qS'_2)$$

Lemma 1.3 *Let $\{P_n\}$ be the monic polynomials orthogonal with respect to the weight (1.1) on the interval $[0, \infty)$. Furthermore, let*

$$R_n = \frac{1}{h_n} \int_0^\infty P_n(y) P_n(y/q) \frac{w(y, \alpha, t; q)}{y} dy,$$

and

$$r_n = \frac{1}{h_{n-1}} \int_0^\infty P_n(y) P_{n-1}(y/q) \frac{w(y, \alpha, t; q)}{y} dy.$$

Then the recurrence coefficients α_n and β_n have the following form

$$q^{2n+\alpha} \alpha_n = \frac{(1 - q^n)}{1 - q} + \frac{1 - q^{n+\alpha+1}}{q(1 - q)} + q^n \left[\frac{t}{q} \right] (R_n + (1 - q) S_{n-1}),$$

$$\beta_n q^{2n-1} = \frac{1}{q^{2\alpha} q^{2n}} \frac{1 - q^n}{1 - q} \frac{1 - q^{n+\alpha}}{1 - q} + \frac{1 - q^n}{q^\alpha} \left[\frac{t}{q} \right] + \frac{q^n}{q^\alpha} \left[\frac{t}{q} \right] r_n + \frac{1}{q^{2\alpha} q^n} \left[\frac{t}{q} \right] S_{n-1}.$$

where $S_{n-1} := \sum_{j=0}^{n-1} R_j$.

Remark 1 *The sum S_n is computed in (3.14) entirely in terms of R_n and r_n . Therefore the lemma above gives expressions for the recurrence coefficients in terms of R_n and r_n only.*

Theorem 1.4 *Let*

$$x_n = \frac{q^{n+\alpha}(1 - q)}{R_n}, \quad y_n = q^n(1 - r_n) \quad \text{and} \quad T = \frac{(1 - q)^2}{q} t.$$

Then the x_n and y_n satisfy the following coupled difference equations

$$(x_n y_n - 1)(x_{n-1} y_n - 1) = q^{2n+\alpha} T \frac{(y_n - 1)(y_n - 1/T)}{(q^n - y_n)},$$

$$(x_n y_n - 1)(x_n y_{n+1} - 1) = -q^{2n+\alpha+1} \frac{(x_n - 1)(x_n - T)}{x_n}. \quad (1.15)$$

Note that these are similar in form to αq -P-IV, [19], and αq -P-V, [4], which we state below:

$$\begin{aligned}
\alpha q - PIV \quad (x_n y_n - 1)(x_{n-1} y_n - 1) &= \frac{(y_n - a)(y_n - b)(y_n - c)(y_n - d)}{(y_n - \kappa \rho_n)(y_n - \rho_n / \kappa)} \\
(x_n y_n - 1)(x_n y_{n+1} - 1) &= \frac{(x_n - 1/a)(x_n - 1/b)(x_n - 1/c)(x_n - 1/d)}{(x_n - \mu w_n)(x_n - w_n / \mu)} \\
\alpha q - PV \quad (x_n y_n - 1)(x_{n-1} y_n - 1) &= q^{2n} \frac{(y_n - a)(y_n - b)(y_n - c)(y_n - d)}{(q^n - \kappa y_n)(q^n - y_n / \kappa)} \\
(x_n y_n - 1)(x_n y_{n+1} - 1) &= q^{2n+1} \frac{(x_n - 1/a)(x_n - 1/b)(x_n - 1/c)(x_n - 1/d)}{(q^{n+1/2} - \mu y_n)(q^{n+1/2} - y_n / \mu)}.
\end{aligned}$$

Here a, b, c, d, κ and μ are parameters, and ρ_n and w_n are in [14]. Furthermore, $abcd = 1$.

This paper is organized as follows. In Section 2 we evaluate the rational functions A_n and B_n in terms of certain auxiliary quantities. In section 3 we give a proof of lemma 1.2 and derive expressions for the recurrence coefficients in terms these auxiliary quantities. In section 4 we give a proof of theorem 1.4. Finally in section 5 we derive a second order, non-linear difference equation, for the quantity $p(n)$.

2 The Structural Relation

In this section we compute the functions $A_n(x)$ and $B_n(x)$ appearing in the relation (1.10) in terms of certain auxiliary quantities. We will make use of the q -product rule and q -integration by parts, namely,

$$\begin{aligned}
D_q(f(x)g(x)) &= f(qx)D_qg(x) + g(x)D_qf(x), \\
\int_0^\infty f(x)D_qg(x) dx &= -\frac{1}{q} \int_0^\infty g(x)D_{q^{-1}}f(x) dx.
\end{aligned}$$

The second formula is valid whenever the integrals

$$\int_0^\infty f(x)g(x)\frac{dx}{x} \quad \text{and} \quad \int_0^\infty f(x)g(qx)\frac{dx}{x}$$

exist. Our first task is to compute the function u for the weight function (1.1). We have,

$$u(x, \alpha, t; q) = \frac{x^2 + \left[\frac{1-q^{-\alpha}}{1-q} \right] qx - q^{1-\alpha}t}{x^2(1 + (q^{-1} - 1)x)}.$$

From this, it follows that,

$$\begin{aligned}
\frac{u(qx, \alpha, t; q) - u(y, \alpha, t; q)}{qx - y} &= \frac{1}{q^\alpha} \left[\frac{t}{qy} \right] \frac{1}{x^2} + \frac{1}{q^\alpha} \left[\frac{q - t(1-q)^2}{q^2(1 + (q^{-1} - 1)y)} \right] \frac{1}{x} \\
&\quad - \frac{1}{q^\alpha} \left[\frac{q - t(1-q)^2}{q^2(1 + (q^{-1} - 1)y)} \right] \frac{1-q}{1 + (1-q)x} - \left[\frac{u(y, \alpha, t; q)}{q} \right] \frac{1}{x}.
\end{aligned} \tag{2.1}$$

Note that (2.1) is a rational function of x and y . Now we look at the effect of the $u(y, \alpha, t; q)$ term in (2.1) on the quantities $A_n(x)$ and $B_n(x)$. First we start with the function $A_n(x)$. For the $u(y, \alpha, t; q)$, term we have

$$\begin{aligned} \frac{1}{h_n} \int_0^\infty u(y, \alpha, t; q) P_n(y) P_n(y/q) w(y, \alpha, t; q) dy &= -\frac{1}{h_n} \int_0^\infty P_n(y) P_n(y/q) D_{q^{-1}} w(y, \alpha, t; q) dy \\ &= \frac{q}{h_n} \int_0^\infty D_q [P_n(y) P_n(y/q)] w(y, \alpha, t; q) dy \\ &= \frac{q}{h_n} \int_0^\infty [P_n(y) D_q P_n(y) + P_n(y) D_q P_n(y/q)] w(y, \alpha, t; q) dy \\ &= 0 \end{aligned}$$

where the last line follows from orthogonality. We repeat the calculation for the $u(y, \alpha, t; q)$ term appearing in $B_n(x)$, and obtain,

$$\begin{aligned} \frac{1}{h_{n-1}} \int_0^\infty u(y, \alpha, t; q) P_n(y) P_{n-1}(y/q) w(y, \alpha, t; q) dy &= - \int_0^\infty P_n(y) P_{n-1}(y/q) D_{q^{-1}} w(y, \alpha, t; q) dy \\ &= \frac{q}{h_{n-1}} \int_0^\infty D_q [P_n(y) P_{n-1}(y/q)] w(y, \alpha, t; q) dy \\ &= \frac{q}{h_{n-1}} \int_0^\infty [P_{n-1}(y) D_q P_n(y) + P_n(y) D_q P_{n-1}(y/q)] w(y, \alpha, t; q) dy \\ &= q \frac{1 - q^n}{1 - q}. \end{aligned}$$

With the following definitions of the auxiliary quantities

$$\begin{aligned} R_n^{(1)} &= \frac{1}{h_n} \int_0^\infty P_n(y) P_n(y/q) \frac{w(y, \alpha, t; q)}{y} dy \\ R_n^{(2)} &= \frac{1}{h_n} \int_0^\infty P_n(y) P_n(y/q) \frac{w(y, \alpha, t; q)}{1 + y(q^{-1} - 1)} dy \\ r_n^{(1)} &= \frac{1}{h_{n-1}} \int_0^\infty P_n(y) P_{n-1}(y/q) \frac{w(y, \alpha, t; q)}{y} dy \\ r_n^{(2)} &= \frac{1}{h_{n-1}} \int_0^\infty P_n(y) P_{n-1}(y/q) \frac{w(y, \alpha, t; q)}{1 + y(q^{-1} - 1)} dy \end{aligned}$$

we find that $A_n(x)$ and $B_n(x)$ appearing in (1.10) are rational functions of x , and read,

$$A_n(x) = \frac{R_n^{(1)}}{x^2} \left[\frac{t}{q} \right] \frac{1}{q^\alpha} + \frac{R_n^{(2)}}{x} \left[\frac{q - t(1 - q)^2}{q^2} \right] \frac{1}{q^\alpha} - (1 - q) \frac{R_n^{(2)}}{1 + x(1 - q)} \left[\frac{q - t(1 - q)^2}{q^2} \right] \frac{1}{q^\alpha} \quad (2.2)$$

$$B_n(x) = \frac{r_n^{(1)}}{x^2} \left[\frac{t}{q} \right] \frac{1}{q^\alpha} + \frac{r_n^{(2)}}{x} \left[\frac{q - t(1-q)^2}{q^2} \right] \frac{1}{q^\alpha} - (1-q) \frac{r_n^{(2)}}{1+x(1-q)} \left[\frac{q - t(1-q)^2}{q^2} \right] \frac{1}{q^\alpha} - \frac{1}{x} \left[\frac{1-q^n}{1-q} \right]. \quad (2.3)$$

We now take note of the fact that the $R_n^{(2)}$ can be expressed in terms of $R_n^{(1)}$. Likewise for $r_n^{(2)}$ in terms of $r_n^{(1)}$. To see this, observe that,

$$q^{-\alpha} \left((1-q) \left[\frac{t}{q} \right] \frac{1}{y} + \left[\frac{q - t(1-q)^2}{q^2} \right] \frac{1}{1 + (q^{-1} - 1)y} \right) w(y, \alpha, t; q) = \frac{1}{q} w(y/q, \alpha, t; q). \quad (2.4)$$

Therefore we have

$$q^{-\alpha} \left((1-q) \left[\frac{t}{q} \right] R_n^{(1)} + \left[\frac{q - t(1-q)^2}{q^2} \right] R_n^{(2)} \right) = q^n. \quad (2.5)$$

$$q^{-\alpha} \left((1-q) \left[\frac{t}{q} \right] r_n^{(1)} + \left[\frac{q - t(1-q)^2}{q^2} \right] r_n^{(2)} \right) = -(1-q)q^{n-1} \sum_{j=0}^{n-1} \alpha_j. \quad (2.6)$$

Remark 2 If we set $t = q/(1 - q^2)$ as was the case in [1] then the coefficients of $R_n^{(2)}$ and $r_n^{(2)}$ in (2.5) and (2.6) vanish, leading immediately to an explicit expression for $R_n^{(1)}$. Substituting this value into the first equation of lemma 1.3 gives an explicit expression for α_n . Subsequently, substituting this expression into (2.6) gives an explicit expression for $r_n^{(1)}$ which can then be substituted into the second equation in lemma 1.3 to give an explicit expression for β_n .

Using (2.5) and (2.6) we now eliminate $R_n^{(2)}$ and $r_n^{(2)}$ from (2.2) and (2.3). Consequently,

$$A_n(x) = \frac{R_n^{(1)}}{x^2} \left[\frac{t}{q} \right] \frac{1}{q^\alpha} + \frac{1}{x} \left(q^n - \frac{1}{q^\alpha} (1-q) \left[\frac{t}{q} \right] R_n^{(1)} \right) - \frac{1-q}{1+x(1-q)} \left(q^n - \frac{1}{q^\alpha} (1-q) \left[\frac{t}{q} \right] R_n^{(1)} \right), \quad (2.7)$$

and

$$B_n(x) = \frac{r_n^{(1)}}{x^2} \left[\frac{t}{q} \right] \frac{1}{q^\alpha} + \frac{1}{x} \left(-(1-q)q^{n-1} \sum_{j=0}^{n-1} \alpha_j - \frac{1}{q^\alpha} (1-q) \left[\frac{t}{q} \right] r_n^{(1)} \right) - \frac{1-q}{1+x(1-q)} \left(-(1-q)q^{n-1} \sum_{j=0}^{n-1} \alpha_j - \frac{1}{q^\alpha} (1-q) \left[\frac{t}{q} \right] r_n^{(1)} \right) - \frac{1}{x} \left[\frac{1-q^n}{1-q} \right]. \quad (2.8)$$

3 The Recurrence Coefficients

In this section we derive expressions for the recurrence coefficients in terms of the quantities $R_n^{(1)}$ and $r_n^{(1)}$. Because we no longer require $R_n^{(2)}$ and $r_n^{(2)}$ we will drop the superscript and use the notation

$$R_n = R_n^{(1)}, \quad r_n = r_n^{(1)} \quad \text{and} \quad S_n = \sum_{j=0}^n R_j.$$

We begin with the derivation of (qS_2') .

Proof of Theorem 1.2

First we write (qS_2) in the form

$$\beta_{n+1}A_{n+1}(x) - \beta_n A_{n-1}(x) = 1 + (x - \alpha_n)(B_{n+1}(x) - B_n(x)) + (1 - q)x B_n(x).$$

If we multiply the above equations by $A_n(x)$ and use (qS_1) to substitute for $(x - \alpha_n)A_n(x)$, we obtain

$$\begin{aligned} \beta_{n+1}A_{n+1}(x)A_n(x) - \beta_n A_n(x)A_{n-1}(x) = \\ A_n(x) + (B_{n+1}^2(x) + u(qx)B_{n+1}(x)) - (B_n^2(x) + u(qx)B_n(x)) \\ + x(1 - q) \left(B_{n+1}(x) \sum_{j=0}^n A_j(x) - B_n(x) \sum_{j=0}^{n-1} A_j(x) \right). \end{aligned}$$

Observe that, up to $A_n(x)$ on the right side, the above is a first order difference equation in n , hence, summing over n , we obtain the q -analog of (S_2')

$$\beta_n A_n(x)A_{n-1}(x) = B_n^2(x) + u(qx)B_n(x) + (1 + (1 - q)x B_n(x)) \sum_{j=0}^{n-1} A_j(x).$$

□

To proceed further, we obtain, equating the coefficients of x^{-2} in (qS_1) , :

$$r_{n+1} + r_n = -\alpha_n R_n + 1. \tag{3.1}$$

Equating the coefficients of x^{-1} in (qS_1) , we obtain the equation :

$$\begin{aligned} - (1 - q) \left(q^n \sum_{j=0}^n \alpha_j + q^{n-1} \sum_{j=0}^{n-1} \alpha_j \right) - \frac{1}{q^\alpha} (1 - q) \left[\frac{t}{q} \right] (r_{n+1} + r_n) - \frac{1 - q^{n+1}}{1 - q} - \frac{1 - q^n}{1 - q} \\ = -\alpha_n q^n - \frac{1 - q^{-\alpha}}{1 - q} + \frac{1}{q^\alpha} \left[\frac{t}{q} \right] (R_n + \alpha_n (1 - q) R_n - (1 - q) S_n - (1 - q)). \end{aligned}$$

The above can be simplified making use of (3.1), and we arrive at

$$\begin{aligned} - (1 - q) \left(q^n \sum_{j=0}^n \alpha_j + q^{n-1} \sum_{j=0}^{n-1} \alpha_j \right) - \frac{1 - q^{n+1}}{1 - q} - \frac{1 - q^n}{1 - q} \\ = -\alpha_n q^n - \frac{1 - q^{-\alpha}}{1 - q} + \frac{1}{q^\alpha} \left[\frac{t}{q} \right] (R_n - (1 - q) S_n). \end{aligned}$$

The above equation simplifies to

$$q^{n+1} \sum_{j=0}^n \alpha_j - q^{n-1} \sum_{j=0}^{n-1} \alpha_j = \frac{1-q^{n+1}}{1-q} + \frac{1-q^n}{1-q} - \frac{1-q^{-\alpha}}{1-q} + \frac{1}{q^\alpha} \left[\frac{t}{q} \right] (qS_n - S_{n-1})$$

We multiply both sides of this equation by the integrating factor q^{n-1} to obtain

$$q^{2n} \sum_{j=0}^n \alpha_j - q^{2n-2} \sum_{j=0}^{n-1} \alpha_j = q^{n-1} \frac{1-q^{n+1}}{1-q} + q^{n-1} \frac{1-q^n}{1-q} - q^{n-1} \frac{1-q^{-\alpha}}{1-q} + \frac{1}{q^\alpha} \left[\frac{t}{q} \right] (q^n S_n - q^{n-1} S_{n-1}).$$

Summing this equation we obtain

$$q^{2n} \sum_{j=0}^n \alpha_j = \frac{1}{q} \left(\frac{1-q^{n+1}}{1-q} \right)^2 - \frac{1}{q} \left(\frac{1-q^{n+1}}{1-q} \right) \left(\frac{1-q^{-\alpha}}{1-q} \right) + \frac{1}{q^\alpha} \left[\frac{t}{q} \right] q^n S_n. \quad (3.2)$$

Remark 3 Because $\sum_{j=0}^n \alpha_j = -p(n+1)$, (3.2) gives the summation $\sum_{j=0}^n R_j$ in a closed form.

From (3.2), we see that,

$$q^{2n} \alpha_n = 2 \left(\frac{1-q^n}{1-q} \right) + \frac{1}{q} - \left(\frac{1}{q} + 1 - q^n \right) \left(\frac{1-q^{-\alpha}}{1-q} \right) + \frac{1}{q^\alpha} \left[\frac{t}{q} \right] q^n (S_n - qS_{n-1})$$

which we write as

$$q^{2n} \alpha_n = 2 \left(\frac{1-q^n}{1-q} \right) + \frac{1}{q} - \left(\frac{1}{q} + 1 - q^n \right) \left(\frac{1-q^{-\alpha}}{1-q} \right) + \frac{1}{q^\alpha} \left[\frac{t}{q} \right] q^n (R_n + (1-q)S_{n-1}). \quad (3.3)$$

Equating the coefficients of $(1+x(1-q))^{-1}$ in (qS_1) we obtain (3.3) again. We go through the same process for (qS'_2) and expect to find three more equations. Equating the coefficients of x^{-4} in (qS'_2) gives

$$\beta_n R_n R_{n-1} = r_n^2 - r_n. \quad (3.4)$$

To proceed further, we equate the coefficients of x^{-2} and x^{-3} in (qS'_2) , which are long formulas. First equating the coefficients of x^{-2} in (qS'_2) produces

$$\begin{aligned} \beta_n \left(q^{2n-1} - 2 \frac{q^{n-1}}{q^\alpha} (1-q) \left[\frac{t}{q} \right] (R_n + qR_{n-1}) \right) = \\ \left(\frac{1}{q^{2\alpha} q^{2n}} \frac{1-q^n}{1-q} \frac{1-q^{n+\alpha}}{1-q} + \left[\frac{t}{q} \right] \frac{1}{q^{2\alpha}} \left(q^\alpha - \frac{2}{q^n} \right) (1-q^n) \right) + \frac{1}{q^{2\alpha} q^n} (2-q^n)(2-q^{n+\alpha}) \left[\frac{t}{q} \right] r_n \\ + \left(\frac{1}{q^{2\alpha} q^n} - 2 \frac{(1-q)^2}{q^{2\alpha}} \left[\frac{t}{q} \right] \right) \left[\frac{t}{q} \right] S_{n-1} + 2 \frac{(1-q)^2}{q^{2\alpha}} \left[\frac{t}{q} \right]^2 r_n S_{n-1}. \end{aligned} \quad (3.5)$$

Now equating the coefficients of x^{-3} in (qS'_2) gives

$$\begin{aligned} \beta_n \frac{1}{q^\alpha} q^{n-1} \left[\frac{t}{q} \right] (R_n + qR_{n-1}) = \\ \frac{1}{q^{2\alpha} q^n} \left[\frac{t}{q} \right] \left(\frac{1-q^n}{1-q} \right) - \frac{1}{q^{2\alpha}} \frac{1}{q^n} \left(\frac{1-q^n + 1-q^{\alpha+n}}{1-q} \right) \left[\frac{t}{q} \right] r_n \\ + \frac{1-q}{q^{2\alpha}} \left[\frac{t}{q} \right]^2 S_{n-1} - \frac{1-q}{q^{2\alpha}} \left[\frac{t}{q} \right]^2 r_n S_{n-1}. \end{aligned} \quad (3.6)$$

We now multiply (3.6) by $2(1-q)$ and add to (3.5) to obtain :

$$\beta_n q^{2n-1} = \frac{1}{q^{2\alpha} q^{2n}} \frac{1-q^n}{1-q} \frac{1-q^{n+\alpha}}{1-q} + \frac{1-q^n}{q^\alpha} \left[\frac{t}{q} \right] + \frac{q^n}{q^\alpha} \left[\frac{t}{q} \right] r_n + \frac{1}{q^{2\alpha} q^n} \left[\frac{t}{q} \right] S_{n-1}. \quad (3.7)$$

Meanwhile (3.6) can be written as

$$\begin{aligned} \beta_n q^{n-1} (R_n + qR_{n-1}) = \frac{1}{q^\alpha q^n} \left(\frac{1-q^n}{1-q} - \left(\frac{1-q^n}{1-q} + \frac{1-q^{n+\alpha}}{1-q} \right) r_n \right) \\ + \frac{1-q}{q^\alpha} \left[\frac{t}{q} \right] (1-r_n) S_{n-1}. \end{aligned} \quad (3.8)$$

In summary, so far we have the 5 equations

$$q^{2n+\alpha} \alpha_n = \frac{(1-q^n)}{1-q} + \frac{1-q^{n+\alpha+1}}{q(1-q)} + q^n \left[\frac{t}{q} \right] (R_n + (1-q)S_{n-1}), \quad (3.9)$$

$$r_{n+1} + r_n = -\alpha_n R_n + 1, \quad (3.10)$$

$$\beta_n q^{2n-1} = \frac{1}{q^{2\alpha} q^{2n}} \frac{1-q^n}{1-q} \frac{1-q^{n+\alpha}}{1-q} + \frac{1-q^n}{q^\alpha} \left[\frac{t}{q} \right] + \frac{q^n}{q^\alpha} \left[\frac{t}{q} \right] r_n + \frac{1}{q^{2\alpha} q^n} \left[\frac{t}{q} \right] S_{n-1}, \quad (3.11)$$

$$\begin{aligned} \beta_n q^{n-1} (R_n + qR_{n-1}) = \frac{1}{q^\alpha q^n} \left(\frac{1-q^n}{1-q} - \left(\frac{1-q^n}{1-q} + \frac{1-q^{n+\alpha}}{1-q} \right) r_n \right) \\ + \frac{1-q}{q^\alpha} \left[\frac{t}{q} \right] (1-r_n) S_{n-1}, \end{aligned} \quad (3.12)$$

$$\beta_n R_n R_{n-1} = r_n^2 - r_n. \quad (3.13)$$

Remark 4 Substituting $t = 0$ into (3.9) and (3.11) gives the recurrence coefficients for the q -Laguerre polynomials. The recurrence coefficients for the Stieltjes-Wigert polynomials can be obtained from (3.9) and (3.11) by setting $\alpha = 0$ and following the steps laid out in remark 2.

Note that both equations (3.11) and (3.12) give an expression for β_n in terms of the auxiliary quantities. Both of these equations are essential because they allow us to eliminate β_n and obtain an expression for the sum S_{n-1} in terms of R_n and r_n only. The sum S_{n-1} is given by the following lemma.

Lemma 3.1 *If $S_n = \sum_{j=0}^n R_j$ then*

$$\begin{aligned} S_{n-1} \left[\frac{t}{q} \right] & \left(\frac{1}{q^{2\alpha} q^n} - \frac{q^n(1-q)(1-r_n)}{q^\alpha R_n} \right) = \\ & - \frac{1}{q^{2n+2\alpha}} \left(\frac{1-q^n}{1-q} \right) \left(\frac{1-q^{n+\alpha}}{1-q} \right) - \frac{q^n}{q^\alpha} \left[\frac{t}{q} \right] r_n \\ & + \frac{1}{q^\alpha R_n} \left(\frac{1-q^n}{1-q} - \left(\frac{1-q^n}{1-q} + \frac{1-q^{n+\alpha}}{1-q} \right) r_n \right) \\ & - q^{2n} \frac{r_n^2 - r_n}{R_n^2} - \frac{1-q^n}{q^\alpha} \left[\frac{t}{q} \right]. \end{aligned} \quad (3.14)$$

Proof

First multiply (3.12) by R_n and use (3.13) to eliminate R_{n-1} . Then substitute for β_n from (3.12) into (3.11). This gives an expression for the sum S_{n-1} in terms of r_n and R_n only. \square

Note that this equation effectively eliminates the sum S_{n-1} from equations (3.9) and (3.11) and consequently we see that α_n and β_n are entirely determined by r_n and R_n .

4 Non-linear Difference equations

In this section we derive the coupled non linear difference equations given in theorem 1.4.

Proof of theorem 1.4

Eliminating α_n from (3.9) and (3.10) we obtain

$$q^{2n+\alpha}(1-r_{n+1}-r_n) = \left(\frac{(1-q^n)}{1-q} + \frac{1-q^{n+\alpha+1}}{q(1-q)} + q^n \left[\frac{t}{q} \right] (R_n + (1-q)S_{n-1}) \right) R_n. \quad (4.1)$$

Eliminating β_n from (3.12) and (3.13) we obtain

$$\begin{aligned} q^{2n+\alpha-1} (r_n^2 - r_n) (R_n + qR_{n-1}) = \\ \left(\frac{1-q^n}{1-q} - \left(\frac{1-q^n}{1-q} + \frac{1-q^{n+\alpha}}{1-q} \right) r_n + q^n(1-q) \left[\frac{t}{q} \right] (1-r_n)S_{n-1} \right) R_n R_{n-1}. \end{aligned} \quad (4.2)$$

We now replace S_{n-1} in (4.1) and (4.2) by the expression given in (3.14), and therefore obtain, respectively

$$\begin{aligned} q^{n-\alpha} \left[(1-r_{n+1}) + \frac{1}{q}(1-r_n) \right] R_n - q^{3n}(1-q)(1-r_{n+1})(1-r_n) \\ = \frac{t}{q^{2\alpha+1}} R_n^3 + \left(\frac{1}{q^{2\alpha+n+1}} \frac{1-q^{2n+\alpha+1}}{1-q} - (1-q)t \frac{q^n}{q^{\alpha+1}} \right) R_n^2 + q^{2n} R_n, \end{aligned} \quad (4.3)$$

and

$$q^{2n-1}(1-q)r_n(1-r_n)^2 - \frac{R_n + qR_{n-1}}{q^{\alpha+1}}r_n(1-r_n) =$$

$$R_n R_{n-1} \left(\frac{1-q}{q^{\alpha+1}} t(1-r_n)(q^n(1-r_n) - 1) + \frac{1-r_n}{q^{2n+2\alpha}} \left[\frac{1-q^{2n+\alpha}}{1-q} \right] - \frac{1}{q^{2n+2\alpha}} \frac{1-q^{n+\alpha}}{1-q} \right). \quad (4.4)$$

Note that (4.3) is a first order difference equation in r_n and a cubic in R_n , whilst (4.4) is the other way around. These equations admit the respective factorizations,

$$\begin{aligned} & [R_n - q^{n+\alpha}(1-q)(q^{n+1} - q^{n+1}r_{n+1})] [R_n - q^{n+\alpha}(1-q)(q^n - q^n r_n)] \\ &= -(1-q)q^n t R_n (R_n - q^{n+\alpha}(1-q)) \left(R_n - \frac{q^{n+\alpha+1}}{t(1-q)} \right) \end{aligned} \quad (4.5)$$

and

$$\begin{aligned} & q^n r_n \left[(q^n - q^n r_n) - \frac{R_n}{q^{n+\alpha}(1-q)} \right] \left[(q^n - q^n r_n) - \frac{R_{n-1}}{q^{n+\alpha-1}(1-q)} \right] \\ &= \frac{t R_n R_{n-1}}{q^\alpha} \left[(q^n - q^n r_n) - \frac{q}{t(1-q)^2} \right] [(q^n - q^n r_n) - 1]. \end{aligned} \quad (4.6)$$

Under the substitutions,

$$x_n = \frac{q^{n+\alpha}(1-q)}{R_n}, \quad y_n = q^n(1-r_n) \quad \text{and} \quad T = \frac{(1-q)^2}{q} t$$

(4.5) and (4.6) are the coupled equations in theorem 1.4. \square

5 Difference equation for $p(n)$

We conclude this paper by stating without proof, a second order, non-linear difference equation, satisfied by the quantity ϱ_n , where

$$\varrho_n := \frac{(1-q)^2}{q} p(n). \quad (5.1)$$

We are happy to provide the relevant PDF file upon request.

First let

$$J_n := \frac{1}{T} \left(1 - q^{-2n-\alpha-1} + \frac{\varrho_n - q\varrho_{n+1}}{1-q} \right), \quad (5.2)$$

and introduce the following variables, defined in terms of J_n and ϱ_n ;

$$F_n := -q^{2n+\alpha} T J_n^2 + [q^{2n+\alpha}(\varrho_n + 1) - 1] J_n - q^n, \quad (5.3)$$

$$G_n := q^{2n+\alpha} (T - \varrho_n) J_n^2 + q^n (1 - q^{n+\alpha}) J_n, \quad (5.4)$$

$$H_n := \frac{J_n J_{n-1} q^n (1 - q^{n+\alpha})}{J_n + J_{n-1}}, \quad (5.5)$$

$$I_n := q^n + \frac{J_n J_{n-1} (1 - q^{2n+\alpha} (1 + \varrho_n))}{J_n + J_{n-1}}. \quad (5.6)$$

After considerable computations, the second order, non-linear difference equation, satisfied by ϱ_n reads,

$$\begin{aligned} & (J_n + J_{n-1})^2 G_n^2 - (J_n + J_{n-1}) \left\{ J_n J_{n-1} q^n (1 - q^{n+\alpha}) (2G_n + F_n^2) \right. \\ & \quad \left. + (q^n (J_n + J_{n-1}) + J_n J_{n-1} [1 - q^{2n+\alpha} (1 + \varrho_n)]) F_n G_n \right\} \\ & \quad + \left\{ q^n (J_n + J_{n-1}) + J_n J_{n-1} [1 - q^{2n+\alpha} (1 + \varrho_n)] \right\} J_n J_{n-1} q^n (1 - q^{n+\alpha}) F_n \\ & \quad + \left\{ q^n (J_n + J_{n-1}) + J_n J_{n-1} [1 - q^{2n+\alpha} (1 + \varrho_n)] \right\}^2 G_n \\ & \quad + (J_n J_{n-1})^2 q^{2n} (1 - q^{n+\alpha})^2 = 0. \end{aligned} \quad (5.7)$$

References

- [1] R. Askey, Orthogonal Polynomials and theta functions, Proceedings of Symposia in Pure Mathematics, **49**, 1989, 299-321.
- [2] E.L. Basor, Y. Chen, Painleve V and the distribution function of a discontinuous linear statistics in the Laguerre unitary ensembles, J. Phys. A **42** (2009) 18pp. 035203
- [3] E.L. Basor, Y. Chen, T. Erhardt, Painleve V and time dependent Jacobi polynomials, J. Phys. A **43** (1) , 015204
- [4] L. Boelen, C. Smet and W. Van Assche, q -Discrete Painleve equations for recurrence coefficients of modified q -Freud orthogonal polynomials, J. Differ. Equ. Appl. **16** (2010) 37-53.
- [5] L. Boelen and W. Van Assche, Variations of Stieltjes-Wigert and q -Laguerre polynomials and their recurrence coefficients, arXiv preprint arXov:1310.3960
- [6] L. Boelen and W. Van Assche, Discrete Painleve equations for recurrence coefficients of semiclassical Laguerre polynomials, Proc. Amer. Math. Soc. **138** (2010) 1317-1331.
- [7] Y. Chen, M.V. Feigin, Painleve IV and degenerate Gaussian unitary ensembles, J. Phys. A **39** (40) (2006) 12381-12393
- [8] Y. Chen and M.E.H. Ismail, Ladder operators and differential equations for orthogonal polynomials, J. Phys. A **30** (22) (1997) 7817-7829.
- [9] Y. Chen and M.E.H. Ismail, Jacobi polynomials from compatibility conditions, Proc. Amer. Math. Soc. **133** (2) (2005) 465-472.

- [10] Y. Chen and M.E.H. Ismail, Ladder operators for q -orthogonal polynomials, *J. Math. Anal. Appl.* **345** (2008) 1-10.
- [11] Y. Chen and A. Its, Painleve III and a singular linear statistics in Hermitian random matrix ensembles I, *Journal of Approximation Theory* **162** (2010), 270-297
- [12] Y. Chen, M. E. H. Ismail and K. A. Muttalib, A solvable random matrix model for disordered conductors, *J. Phys.: Cond. Matt.* **4** (1992) L417; K. A. Muttalib, Y. Chen, M. E. H. Ismail, and V. N. Nicopoulos, New family of unitary random matrices, *Phys. Rev. Lett.* **71** (1993) 471-475.
- [13] G. Filipuk and C. Smet, On the recurrence coefficients for Generalized q -Laguerre Polynomials, *Journal of Nonlinear Mathematical Physics*, **20** (2013), Supplement 1, 48-56
- [14] B. Grammaticos and A. Ramani, "Discrete Painleve Equations: A Review," *Springer Lecture Notes in Physics*, **644** (2004) 245-321.
- [15] M.E.H. Ismail and Z.S.I. Mansour, q -Analogues of Freud weights and nonlinear difference equations, *Advances in Applied Mathematics*, **45** (2010) 518-547.
- [16] M.E.H. Ismail, Sarah Jane Johnstone and Z.S.I. Mansour, Structure relations for q -polynomials and some applications **90**, Nos. 3-4, March-April 2011, 747-767
- [17] A. Magnus, Painleve-type differential equations for the recurrence coefficients of semi-classical orthogonal polynomials, *J. Comput. Appl. Math.* **57** (1-2) (1995) 215-237.
- [18] Daniel S. Moak, The q -analog of the Laguerre polynomials, *J. Math. Anal. Appl.* **81** (1981) 20-47.
- [19] W. Van Assche, Discrete Painleve equations for recurrence coefficients of orthogonal polynomials, *Proceedings of the International Conference on Difference Equations, Special Functions and Orthogonal Polynomials*, World Scientific (2007), 687-725.